When $p = \tau = 0$, conditions (4.9), (4.11) take the form

$$D\cos^2\theta = 0,\ D\sin\theta\cos\theta = 0$$
$$D + \frac{c(\lambda_- + 2\mu_-)}{6\mu_- K_-}\sigma_+\sin^2\theta = 0 \tag{4.12}$$
$$D \equiv -\frac{1}{6}\left[\frac{\lambda}{\mu K}\right]_-^+\sigma_+ - \frac{c(\lambda_- + \mu_-)}{3\mu_- K_-}\sigma_+\sin^2\theta - \delta$$

System (4.12) has the following solutions:

$$1)\ \sin\theta = 0,\quad \sigma_+ = -6\delta\Big/\left[\frac{\lambda}{\mu K}\right]_-^+\ (= \sigma_-)$$
$$2)\ \cos\theta = 0,\quad \sigma_+ = -6\mu_+\delta\left(\frac{\lambda_+}{K_+} + \frac{\lambda_-}{K_-}\right)^{-1}\ \left(= \frac{\mu_+}{\mu_-}\sigma_-\right)$$

Thus we have for the first(second) solution the corresponding interphase planes perpendicular (parallel) to the x^1 axis.

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CONFIGURATIONAL FORCES IN THE MECHANICS OF A SOLID DEFORMABLE BODY*

G.P. CHEREPANOV

A configurational force /1—3/, which always originates in a deformable solid whenever the stress source moves, represents physically the contribution of the external strain and stress fields to the dissipation of energy, taken per unit path length of the source. When the stress source (singularity) is internal, the configurational force is the fundamental parameter controlling the process of motion and it can be called a driving force. Linear singularities of the type of crack and dislocation contours, point singularities of the type of small cavities and inclusions, etc. are examples of each cases. If the singularity is generated directly by external forces, the configurational force plays an auxiliary role and such cases will be examined below. This is the problem of the motion of a small solid body over the surface of a half-space, and different schemes of wedge motion in an unbounded elasto-plastic space.

1. Motion of a small solid over the surface of a half-space. Let a concentrated force $(T, 0, -N)$ move at a constant velocity V over the surface of a solid half-space $z < 0$ (Fig.1), stretched by a stress σ_x^∞. Its surface is considered to be free of external loads, with the exception of the point O moving with the velocity V of the origin. Since the field of quasistatic stresses and strains in a solid is stationary in the $Oxyz$ coordinate system, the following equality /1—3/ holds for any materials for any finite deformations:

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$$\Gamma = \int_\Sigma (Un_x - \sigma_{ij} n_j u_{i,x})\, d\Sigma \quad (i, j = 1, 2, 3) \tag{1.1}$$

Here Γ is the configurational flux of solid body energy dissipation into the origin, Σ is an arbitrary surface in the lower half-space including the origin (the contour of the edge Σ lies on the boundary $z = 0$), σ_{ij} and u_i are stress and displacement components, respectively, n_i are components of the direction of the external normal to the surface Σ ($n_x = n_1$, $u_{i,x} = u_{i,1}$), and U is the strain energy per unit volume.

Evaluating Γ over the surface of a parallelepiped $|x_i| \leqslant R$ $(i = 1, 2)$, $0 < x_3 < \delta$ as $\delta \to \infty$, $R \to \infty$, $\delta/R \to 0$, we obtain the configurational force /1–3/

$$\Gamma = T u^\infty_{x,x} \tag{1.2}$$

(the quantity $u^\infty_{x,x}$ is a function of σ_x^∞ and the time t determined by the rheological equation of state).

Indeed, as $\delta/R \to 0$ the contribution to Γ from the small sides of the rectangle will be infinitesimal, while we have $n_x = n_y = 0, n_z = -1$ (Fig.1) on the large side. Consequently, according to (1.1) we will have (below the integration is carried out between the limits $-R$ and $+R$)

$$\Gamma = \lim_{R\to\infty,\, \delta/R\to 0} \iint (\sigma_{xz} u_{x,x} + \sigma_{yz} u_{y,x} + \sigma_{zz} u_{z,x})|_{z=-\delta}\, dx\, dy$$

By the rule of Γ–integration /2, 3/, we have

$$\Gamma = \lim_{R\to\infty,\, \delta/R\to 0} \iint (\sigma^s_{xz} u^\infty_{x,x} + \sigma^\infty_{xz} u^s_{x,x} + \sigma^s_{yz} u^\infty_{y,x} + \sigma^\infty_{yz} u^s_{y,x} + \sigma^s_{zz} u^\infty_{z,x} + \sigma^\infty_{zz} u^s_{z,x})|_{z=-\delta}\, dx\, dy$$

The superscripts s and ∞ refer to the intrinsic field of singularities and to the unperturbed field, respectively.

Because of the boundary conditions

$$u^\infty_{y,x} = u^\infty_{z,x} = 0, \quad \sigma^\infty_{xz} = \sigma^\infty_{yz} = \sigma^\infty_{zz} = 0, \quad \iint \sigma^s_{xz}\, dx\, dy = T$$

We hence obtain (1.2).

The total energy dissipation D per unit concentrated force path length is

$$D = T + T u^\infty_{x,x} = T\,(1 + u^\infty_{x,x}) \tag{1.3}$$

Let the action of the force on the half-space be transmitted through some small rigid body B; then the force T will equal the resistance experienced by the body B. (At the same time, the quantity T equals the work performed by the body B along unit path. We emphasize that relationship (1.3) is not the energy-conservation equation).

For small N and $u^\infty_{x,x} = 0$ the quantity $D = k + \mu_f N$, where k and μ_f are the cohesion and friction coefficients, respectively (Coulomb's law). Hence, in conformity with (1.3) we obtain

$$T = (k + \mu_f N)\,(1 + u^\infty{}_x) \tag{1.4}$$

This is a generalized Coulomb law that takes account of the influence of the deformation of the half-space by secondary forces that can be substantial for materials capable of experiencing significant deformation prior to rupture (low-modulus polymers and composities, certain polymers and metals at elevated temperatures). Formally, if the old Coulomb formula is used, the effect of the configurational force results in a decrease in k and μ_f as the half-space is stretched (and an increase in k and μ_f for compression).

Upon indentation of the body B in a half-space with substantial plastic deformation ("ploughing"), it is natural to make the following assumption: the energy dissipation D is directly proportional to the degree of identation h, i.e., $D = \lambda h$, where λ is a certain constant of thesystem that is dependent on the velocity V and on the shape of the indentor. The magnitude of the indentation h as a function of the normal force N is determined experimentally for any given indentor. Consequently, the fundamental relation for ploughing can be written thus $T = \lambda h\,(N)$. For small N and constant area of contact, the function $h\,(N)$ is linear. This limit case corresponds to the classical Coulomb law.

As an example of ploughing we consider the plane problem of cutting chips of thickness h from an ideal elasto-plastic half-space; the body B is a rigid cutter in the shape of an angle α (Fig.2). We approximate the free surface of the chip near the contact area by a circle of radius r_0 and we consider the whole chip in the neighbourhood of this area to be in the plastic state. The stress field in the plastic domain (and on the contact area) will be the following /4/:

$$\sigma_r = -2\tau_s \ln\frac{r}{r_0}, \quad \sigma_\theta = -2\tau_s \ln\frac{r}{r_0} - 2\tau_s, \quad \tau_{r\theta} = 0 \tag{1.5}$$

(τ_s is the shear yield point r, θ are a polar coordinate system with centre O for the

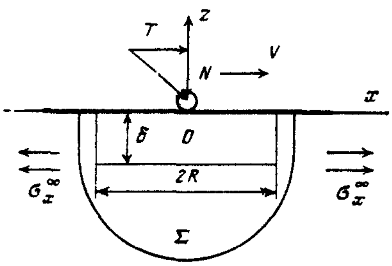

Fig.1

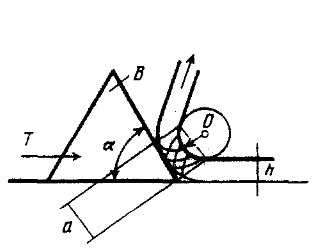

Fig.2

approximating circle). We shall consider approximately that the tangential stresses are zero on the contact area while the normal pressure is the stress σ_r in the plastic domain for $r = r_0 + h$. We hence obtain

$$T = 2a\tau_s \sin\alpha \ln(1 + h/r_0) \tag{1.6}$$

The length a of the contact area and the chip radius r_0 are directly proportional to h, where the proportionality factors can only depend on $\alpha, \tau_s/E$ and ν, where α is the cutting angle, E and ν are Young's modulus and Poisson's ratio (on the basis of a dimensional analysis). Since $D = T$ for $u^\infty_{x,x} = 0$, then the example presented confirms and clarifies the assumption that $D \sim h$.

Therefore, the following relation is found: the resistive force acting on the cutter by the material is directly proportional to the depth of the cut (the chip thickness), the shear strength of the material and the sine of the slope of the cutter plane. This relation can be utilized for the optimal design of teeth on a drill bit (for instance, to select the optimal magnitude and slope of the teeth in the most promising tooth construction "stratapax").

2. Cutting an elasto-plastic body.

Let a semi-infinite rigid rectangular tool to thickness $2h$ move at constant velocity V in an infinite space of ideal elasto-plastic material (plane deformation, Fig.3). At infinity the space is considered to be subjected to the action of the stresses σ_x^∞ and σ_y^∞ ($\sigma_y^\infty < 0, \tau_{xy}^\infty = 0$). For simplicity, the sides of the tool will be considered to be smooth, i.e., the tangential friction stress on the side surface is zero. (The velocity of motion is small compared with the speed of sound).

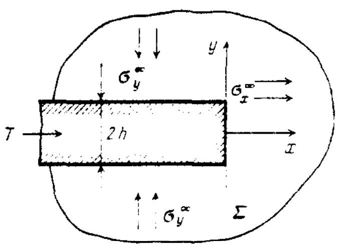

Fig.3

For the stationary stress and strain fields under consideration, the following equality holds /1/:

$$\int_\Sigma (Un_x - \sigma_{ij}n_j u_{i,x})\, d\Sigma = 0 \quad (i, j = 1, 2) \tag{2.1}$$

Here Σ is a closed contour consisting of the cutter surface contour Σ_c and an arbitrary arc Σ_∞ endircling the tool; the moving rectangular Cartesian coordinate system xy is coupled to the tool (the x axis is the axis of symmetry of the problem). We recall that (2.1), exactly as (1.1), is valid for any rheological model of the body.

The following hold:

$$\int_{\Sigma_c} (Un_x - \sigma_{ij}n_j u_{i,x})\, d\Sigma = D - T \tag{2.2}$$

$$\Gamma = \int_{\Sigma_\infty} (Un_x - \sigma_{ij}n_j u_{i,x})\, d\Sigma = -Tu^\infty_{x,x} - 2h\sigma_y^\infty \tag{2.3}$$

where T is the external force applied to the cutter in the direction of the x axis, D is the total dissipation of energy per unit path length of the cutter, and Γ is the configurational force (the contribution of the external field to the dissipation D).

On the side faces of the cutter $n_x = 0, u_{y,x} = 0, \tau_{xy} = 0$, consequently only the frontal surface of the cutter can be considered as Σ_c. In this case of an elasto-plastic body, the quantity U on the frontal surface equals the energy dissipated per unit volume (the dissipative function) since the elastic component is negligibly small because of the finiteness of the deformation. Consequently, the first component on the left side of (2.2) yields D. Furthermore, because of the equality of the normal velocity components, on the frontal surface we have $V = \partial u_x(x_{10} - Vt)/\partial t = -V\partial u_x/\partial x$ (where $x_{10} = x + Vt$ is the coordinate associated with fixed space at infinity), i.e., $u_{x,x} = -1$. Moreover, the quantity $\tau_{xy}u_{y,x}$ will be an odd function of y on the frontal surface. Hence (2.2) results. It can also be proved that (2.2) is valid when taking account of arbitrary friction on the side faces of the cutter and when taking account of the elastic component U on the side parts.

We take the contour of the rectangle $|x| \leqslant R, |y| \leqslant \delta$ as $R \to \infty, \delta \to \infty, \delta/R \to 0$ as Σ_∞. Then the contribution of the integral over the small sides of the rectangle will be negligible, and will be $n_x = 0, n_y = 1$ on the sides $y = \pm\delta$, and by using the rule of Γ-integration /2, 3/, we will have (the integration is performed below within the limits $-R$ to $+R$)

$$\Gamma = 2\int (\sigma_{ij}^{\infty} n_j u_{i,x}^s + \sigma_{ij}^s n_j u_{i,x}^{\infty})\,|_{y=\delta}\, dx = 2\left(\sigma_y^{\infty}\int u_{y,x}^s\, dx + u_{i,x}^{\infty}\int \sigma_{ij}^s n_j\, dx\right)$$

The unperturbed (i.e., no wedge) and perturbed stress and strain components are denoted, respectively, by the superscripts ∞ and s. (This decomposition evidently does not contain any assumptions; it simply defines the perturbed component as the difference between the total and unperturbed quantities.) Furthermore, we use the equilibrium conditions and the presence of a displacement jump $\Delta u_y = 2h$ in the body during traversal of any contour Σ_∞. We have

$$\lim_{R\to\infty}\int u_{y,x}^s\, dx = -h, \quad \lim_{R\to\infty}\int \sigma_{ij}^s n_j\, dx = -T\delta_{1i}$$

Relationship (2.3) is proved.

On the basis of (2.1)−(2.3) we obtain

$$D = T + Tu_{x,x}^{\infty} + 2h\sigma_y^{\infty} \tag{2.4}$$

This relationship could also have been obtained by the method indicated in Sect.1, by using the value of the Γ-residue for the dislocations and the concentrated force /1−3/.

We hence find the following expression for the magnitude of the cutter resistance force:

$$T = (D - 2h\sigma_y^{\infty}) / (1 + u_{x,x}^{\infty})$$

In the case when $\sigma_y^{\infty} > 0$ the tension at infinity results in separation of the material from the side faces at a certain distance from the cutter frontal section; the assignment of σ_y^{∞} becomes incorrect. We assume that for sufficiently large r the material is linearly elastic with Young's modulus E and Poisson's ratio ν (μ is the shear modulus). In this case, the stress intensity factor of the external field K_I, equal to $\lim (\sigma_y \sqrt{2\pi r})$ as $r \to \infty$, should be given at infinity (r is the distance from the frontal part of the wedge). The following expression for the resistance force is obtained in the same way:

$$T = \frac{ED - (1-\nu^2) K_I^2}{E(1 + u_{x,x}^{\infty})} \quad (K_I > 0, K_{II} = K_{III} = 0)$$

since the magnitude of the configurational force will equal

$$\Gamma = -Tu_{x,x}^{\infty} - (1-\nu^2) E^{-1} K_I^2 \tag{2.5}$$

in this case instead of (2.3).

When there is no external field (when the configurational force equals zero) $D = T$. This relationship enables D to be determined (by test or from model theory) as a function of h and the physical properties of the material.

Depending on the relative role of elasticity and plasticity, as well as on the sign of the stress σ_y, the following six fundamental versions of the flow of an elasto-plastic material around a cutter (Fig.4) can be extracted for large r.

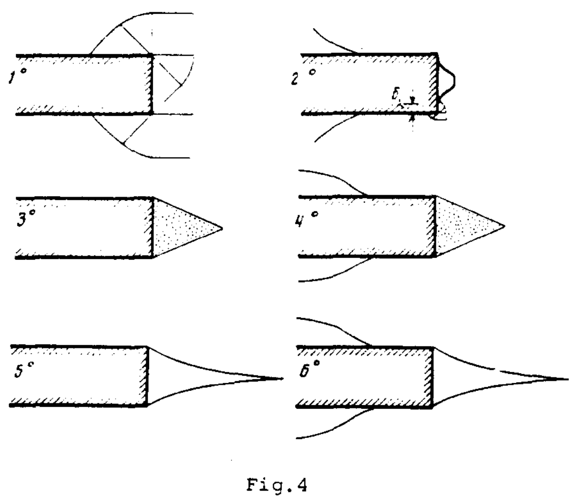

Fig.4

Version 1°. For $K_{Ic} \gg \mu\sqrt{h}$ and $\sigma_y^\infty \leqslant 0$, where K_{Ic} is the fracture toughness, the role of the elasticity forces near the cutter end is negligibly small. In the case of an incompressible, ideal, elasto-plastic Tresca-Saint Venant material, the family of characteristics in the plastic domain near the end of the cutter can be taken as for a plastic wedge with aperture angle $3\pi/2$. In this case, according to the theory of plasticity, we have on the frontal section /4/

$$\sigma_x = -2\tau_s(1+\pi) - q, \quad \sigma_y = -2\pi\tau_s - q, \quad \tau_{xy} = 0 \tag{2.6}$$

(τ_s is the shear yield point, q is the magnitude of the side support for $y = \pm h$ near the cutter angles determined from solutions of the elasto-plastic problem in the large).

Version 2°. For $K_{Ic} \gg \mu\sqrt{h}$ and $K_I > 0$ a symmetric cavity appears on the frontal face in addition to the cavities on the side faces of the cutter. The presence of this cavity is due to the finiteness of the plastic deformations in the tip zone and the fact that for $K_I > 0$ any fracture in the plastic body would have parabolically rounded-off ends /1/ (the inflow, i.e., the stretching contact pressure between the cutter and the material is eliminated). In this case, the stresses on the frontal area of contact will be $\sigma_x = -2\tau_s$, $\sigma_y = \tau_{xy} = 0$ for a well-developed frontal cavity so that the resistance force is $T = 4\delta\tau_s$, where δ is the width of the area (determined from the solution of the elasto-plastic problem in the large).

Version 3°. For $K_{Ic} \sim \mu\sqrt{h}$, $\sigma_y^\infty \leqslant 0$ the influence of elastic cleavage is substantial so that an elastic dead zone filled by the very same material moving at the speed of the cutter is formed in front of the frontal face of the cutter (shown by dots in Fig.4). The solution of this problem can be found in a "quasibrittle" approximation, i.e., by considering the plastic domain surrounding the dead zone to be a sufficiently thin layer along its boundary.

Version 4°. For $K_{Ic} \sim \mu\sqrt{h}$ and $K_I > 0$, symmetry cavities appear in addition to the dead zone in the preceding version, along the side faces of the cutter. In this case the quasibrittle approximation also enables an analogous effective investigation to be made.

Versions 5° and 6°. For $K_{Ic} \ll \mu\sqrt{h}$ the influence of plasticity is small compared with the elastic cleavage: a gaping quasibrittle normal rupture crack is formed in front of the cutter, and cavities still appear along the sides of the wedge for $K_I > 0$. (These cases are the limits for Versions 3° and 4°, respectively).

Because of the greatest practical significance of Versions 3° and 4°, a solution of the corresponding problems is given below.

One of the most important results of the general qualitative analysis presented for the fracture process during cutting is the deduction about the substantial influence of the prestress of the item or specimen being treated: preliminary tension facilitates cutting considerably while compression makes it difficult. This result is a consequence of the influence of the configurational force - it is most substantial for internal cutting; however, it can also be significant for surface cutting if an external force is applied to the chip being stripped off (for instance, by using a fluid or gas jet). The effect obtained can be used in designing the optimal structure of a cutting instrument.

3. The problem of cleavage with a dead zone under compression. Let a smooth absolutely rigid semi-infinite thin wedge be along the negative half-axis x, and for $y = 0$, $0 < x < l$ ahead of it let there be a fine cavity, a dead zone filled with fragments of fractured material (the plane problem, Fig.5). At infinity the space is compressed by the stress $\sigma_y = -q$ $(q > 0)$. The action of the dead zone on the edge of the cavity will be simulated by a constant pressure $p > 0$ and a tangential friction load $\tau > 0$. Because of the symmetry of the problem we can limit ourselves to the upper half-plane $y > 0$.

We hence obtain the following boundary value problem of the plane theory of elasticity for the half-plane $y > 0$:

$$\begin{gathered} y = 0, \quad x < 0, \quad v = h, \quad \tau_{xy} = 0 \\ y = 0, \quad 0 < x < l, \quad \sigma_y = -p, \quad \tau_{xy} = \tau \\ y = 0, \quad x > l, \quad v = 0, \quad \tau_{xy} = 0 \\ x^2 + y^2 \to \infty, \quad \sigma_y = -q, \quad \sigma_x = \tau_{xy} = 0 \end{gathered} \tag{3.1}$$

We use the Kolosov-Muskhelishvili representation of plane elasticity theory

$$\begin{gathered} \sigma_x + \sigma_y = 2[\Phi(z) + \overline{\Phi(z)}] \quad (z = x + iy) \\ \sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)] \\ 2\mu(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \\ \varphi'(z) = \Phi(z), \quad \psi'(z) = \Psi(z) \end{gathered} \tag{3.2}$$

Here $\Phi(z)$ and $\Psi(z)$ are analytic functions of the complex variable z in the upper half-plane, and u, v are displacements.

In this problem there are two singularities in addition to $z = \infty$: $z = 0$ and $z = l$.

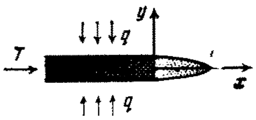

Fig.5

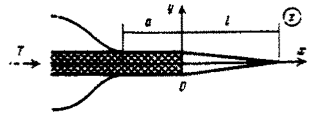

Fig.6

The point $z = l$ is the tip of the separation crack; the function $\Phi(z)$ is of order $(z-l)^{-1/2}$ there. In the neighbourhood of the other point the elastic body is tangent to the frontal face of the cutter near the angular point when the dead zone is filled insufficiently with fragmentary material, which results in infinite stresses near this point (of order $x^{-1/2}$), to rough working of the body and further filling of the dead zone with fracture products. The rough working process ceases when the pressure of the fragments in the dead zone becomes sufficiently high in order to separate the cavity walls near the angles and to reduce the stress concentration to a minimum near $z = 0$. We consider the latter, stationary, stage of the rough-working process by requiring that the desired function $\Phi(z)$ should not have a power-law singularity at the point $z = 0$.

By using the representation (3.2) and the theory of singular integral equations, the solution of the boundary value problem (3.1) in the above class of functions can be reduced to the following form:

$$\Phi(z) = \frac{1}{4}q - \frac{1}{2}p - \frac{\tau}{2\pi}\ln\left(1-\frac{l}{z}\right) + \frac{1}{2}(p-q)\sqrt{\frac{z}{z-l}} \tag{3.3}$$

$$z\Phi'(z) + \Psi(z) = -\frac{1}{2}q + \frac{\tau}{\pi}\ln\left(1-\frac{l}{z}\right)$$

$$\left(\sqrt{z/(z-l)} \to 1 \quad \text{as} \quad z\to\infty\right)$$

By using (3.2) and (3.3) we find the shape of the cavity ahead of the cutter (for $y = +0,\ 0 < x < l$)

$$v = \frac{\varkappa-1}{4\mu}\tau(x-l) + \frac{\varkappa+1}{4\mu}(p-q)\left[-\sqrt{x(l-x)} + l\,\mathrm{arctg}\sqrt{\frac{l-x}{x}}\right] \tag{3.4}$$

For $x = 0$ the displacement v should be equal h

$$p - q = \frac{8\mu h}{\pi l(\varkappa+1)} - \frac{2}{\pi}\frac{\varkappa-1}{\varkappa+1}\tau \tag{3.5}$$

Since $\tau = p\mu_f$, where μ_f is the coefficient of friction, we obtain the pressure p in the stationary dead zone from (3.5):

$$p = \left[\frac{8\mu h}{\pi l(\varkappa+1)} + q\right]\left(1 - \frac{2}{\pi}\mu_f\frac{\varkappa-1}{\varkappa+1}\right)^{-1} \tag{3.6}$$

According to (3.3) and (3.5)

$$\Phi(z) = \frac{4\mu h + 2\tau l}{2\pi z(\varkappa+1)} \quad \text{as} \quad z\to\infty$$

This corresponds to a wedge dislocation of power $2h$ and force $2\tau l$ at infinity.

The stress σ_x in the dead zone is determined from the equilibrium equation

$$d(v\sigma_x)/dx = -p\,\partial v/\partial x + \tau$$

Since we will have $\sigma_x = 0$ for $x = l$, we hence obtain $\sigma_x = -p + \tau(x-l)/v(x)$. Therefore, the pressure on the frontal face of the wedge is

$$-\sigma_x = p + \tau l/h = p(1 + \mu_f l/h) \tag{3.7}$$

Consequently, the frontal resistance force T to the wedge motion, in conformity with (3.6) and (3.7), will equal

$$T = 2h\left(1+\mu_f\frac{l}{h}\right)\left[\frac{8\mu h}{\pi l(\varkappa+1)} + q\right]\left(1-\frac{2}{\pi}\mu_f\frac{\varkappa-1}{\varkappa+1}\right)^{-1} \tag{3.8}$$

By using (3.2) and (3.3) we find the stress intensity factor $K_I = \sqrt{2\pi l}\,(p-q)$ and the length of the cavity in front of the cutter from the brittle fracture criterion and equations (3.6)

$$l = \frac{4a^2}{\pi\left(b+\sqrt{b^2-4ac}\right)^2} \tag{3.9}$$

$$\left(a = \frac{8\mu h}{\varkappa+1},\ b = \frac{1}{\sqrt{2}}K_{Ic}\left(1-\frac{2}{\pi}\frac{\varkappa-1}{\varkappa+1}\mu_f\right), \qquad c = \frac{2}{\pi}\frac{\varkappa-1}{\varkappa+1}q\mu_f\right)$$

In particular, for $q=0,\ \nu={}^1/_2,\ \varkappa=1$ we will have $l=32\pi^{-1}\mu^2h^2K_{1c}^{-2}$ and it is evident that for $K_{Ic}<3\mu\sqrt{h}$ the required condition $l\gg h$ is satisfied fairly well.

4. The problem of cleavage with a dead zone under tension. In the case of tension of the space across the wedge, the material breaks away from the side faces of the wedge, and symmetric empty cavities (Fig.6) form along $y=0$ for $x<-a$. In this case, the stress intensity factor at infinity should be given as $K_I=K_I^\infty>0$. This problem belongs to the class N in which the Saint Venant principle is not satisfied /1/.

The following boundary value problem holds for the half-plane $y>0$ with the other assumptions as before:

$$\begin{aligned}
&y=0,\quad x<-a,\quad \sigma_y=\tau_{xy}=0\\
&y=0,\quad -a<x<0,\quad v=h,\quad \tau_{xy}=0\\
&y=0,\quad 0<x<l,\quad \sigma_y=-p,\quad \tau_{xy}=\tau\\
&y=0,\quad x>l,\quad v=0,\quad \tau_{xy}=0\\
&y=0,\quad x\to\infty,\quad \sigma_x=\sigma_y=K_I^\infty/\sqrt{2\pi x},\quad \tau_{xy}=0
\end{aligned}\tag{4.1}$$

From physical considerations in the conditions of the stationary stage of the process, it follows that the function $\Phi(z)$ will behave as follows at the singular points of this boundary value problem:

$$\begin{aligned}
&z\to-a,\quad z\to 0,\quad \Phi(z)=O(1);\\
&z\to l,\quad \Phi(z)=K_I/[2\sqrt{2\pi(z-l)}]
\end{aligned}\tag{4.2}$$

Here K_I is the stress intensity factor at the end of the dead zone (a crack with a filler).

By using (3.2) the solution of the boundary value problem (4.1) in the class of functions (4.2) can be reduced to the following form

$$\begin{aligned}
&z\Phi'(z)+\Psi(z)=\frac{\tau}{\pi}\ln\left(1-\frac{l}{z}\right)\\
&\Phi(z)=\frac{1}{\pi i}X(z)\int_L\frac{f(x)\,dx}{X^+(x)(x-z)}\\
&X(z)=\sqrt{z\frac{z+a}{z-l}},\quad f=-\frac{\tau}{2\pi}\ln\left|1-\frac{l}{x}\right|+\begin{cases}0, & x<-a\\ -{}^1/_2p, & 0<x<l\end{cases}\\
&(L:y=0,-\infty<x<-a\ \text{ plus }\ y=0,0<x<l)\\
&\quad X(z)\to|\sqrt{x}|\quad\text{as}\quad z\to\infty+i0
\end{aligned}\tag{4.3}$$

Here $X^+(x)$ denotes the value of $X(z)$ on the upper edge of the slit.

Two conditions that give the behaviour of $\Phi(z)$ as $z\to\infty$ and $z\to l$, equation (3.7), the equation $\tau=p\mu_f$, and the condition

$$y=0,\quad \int_0^l\frac{\partial v}{\partial x}dx=h\tag{4.4}$$

determine the unknown parameters of the problem, a,l,p,τ,T.

By analysing formulas (3.4), (3.6) and (3.9), we see that for real values of μ_f the quantities p,l and $v(x)$ depend fairly weakly on τ (the exception is the frontal resistance force T). Consequently, for simplicity we neglect τ when calculating a,l,p by assuming $\tau=0$. According to (3.8), the quantity T as before equals

$$T=2ph\,(1+\mu_fl/h)\tag{4.5}$$

where we find p and l approximately.

By using (3.6) and (4.4) for $\tau=0$, we find

$$y=0,\quad 0<x<l,\quad \frac{\partial v}{\partial x}=\frac{(\varkappa+1)p}{4\mu\pi}X^+(x)\int_0^l\frac{dt}{X^+(t)(t-x)}\tag{4.6}$$

Moreover, according to (4.4) we have for $\tau=0$

$$\begin{aligned}
&z\to\infty,\quad \Phi(z)=\frac{p}{2\pi i\sqrt{z}}\int_0^l\frac{dx}{X^+(x)}\\
&z\to l,\quad \Phi(z)=\frac{p}{2\pi}\left[\frac{l(a+l)}{z-l}\right]^{1/2}\int_0^l\frac{dx}{|[x(x+a)(x-l)]^{1/2}|}
\end{aligned}\tag{4.7}$$

By using (4.6) and (4.7), conditions (4.2), (4.3) and (4.5) assist in obtaining the following equations to find a,l, and p (all the radicals are positive):

$$\int_0^l\left[\frac{l-x}{x(x+a)}\right]^{1/2}dx=\frac{1}{p}\sqrt{\frac{\pi}{2}}K_I^\infty\tag{4.8}$$

$$\int_0^l \frac{dx}{[x(x+a)(l-x)]^{1/2}} = \sqrt{\frac{\pi}{2}}\,\frac{K_{Ic}}{p\,[l\,(l+a)]^{1/2}} \tag{4.9}$$

$$\int_0^l \left[\frac{x\,(x+a)}{l-x}\right]^{1/2} dx \int_0^l \left[\frac{l-t}{t\,(t+a)}\right]^{1/2} \frac{dt}{t-x} = \frac{4\pi\mu h}{p\,(\varkappa+1)} \tag{4.10}$$

5. Cleavage with a yawning separation crack. When the fracture products are removed from the crack cavity (or at the beginning of cleavage), the influence of the filler can be neglected. In this case, $p = \tau = 0$ should be assumed in formulating the boundary value problems (3.1) for compression, and (4.1) for tension, and the solution of the boundary value problems should be sought in the class of functions having an integrable singularity as $z \to 0$. We limit ourselves to the case of compression. The solution of the boundary value problem will be

$$\Phi(z) = \frac{-qz + C}{2\sqrt{z(z-l)}} + \frac{1}{4}q, \quad z\Phi'(z) + \Psi(z) = -\frac{1}{2}q \tag{5.1}$$

($\sqrt{z(z-l)} \to z$ as $z \to \infty$, and C is a real constant).

On the basis of (3.2) and (5.1), we find the shape of the crack cavity for $y = +0,\ 0 < x < l$

$$v = \frac{\varkappa+1}{4\mu}\left[(2C - ql)\left(\frac{\pi}{2} - \operatorname{arctg}\sqrt{\frac{x}{l-x}}\right) - q\sqrt{x(l-x)}\right] \tag{5.2}$$

Since $v = h$ for $x = 0$, we obtain

$$2C - ql = \frac{8\mu h}{\pi(\varkappa+1)} \tag{5.3}$$

According to (3.2) and (5.1) we have

$$K_1 = \sqrt{2\pi/l}\,(C - ql), \quad C - ql = K_{1c}\sqrt{l/(2\pi)} \tag{5.4}$$

We find from (5.3) and (5.4)

$$l = \frac{K_{Ic}^2}{2\pi q^2}\left[\left(1 + \frac{16q\mu h}{(\varkappa+1)K_{Ic}^2}\right)^{1/2} - 1\right]^2 \tag{5.5}$$

$$C = \frac{4\mu h}{\pi(\varkappa+1)} + \frac{1}{2}ql \quad (q > 0)$$

In particular, for $q = 0$

$$l = \frac{32\mu^2 h^2}{\pi K_{Ic}^2 (1+\varkappa)^2}, \quad C = \frac{4\mu h}{\pi(\varkappa+1)}$$

The singularities $z = 0$, $z = l$, and $z = \infty$ are unique sources (for $z = \infty$) and sinks (for $z = 0$ and $z = l$) of elastic energy since the wedge is ideally smooth and the body is ideally elastic.

We examine the invariant Γ-integral (2.1) over the contour displayed in Fig.7. By using (5.1) and (3.2) we can find

$$T = (1 - \nu^2)\,E^{-1}K_{Ic}^2 + 2qh \qquad (q > 0) \tag{5.6}$$

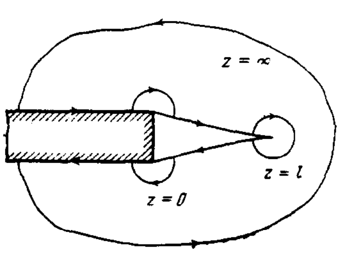

Fig.7

Here T is the sum of Γ-residues for traversal around the angles of the wedge (for $z = 0$). The first term is the Γ-residue for by passing the point $z = l$ and the second is the Γ-residue for bypassing the infinitely remote point.

On the basis of (5.6), a resistance force T acts on the wedge as it moves. This is a simple and instructive example of a configurational force! Let us note /5—10/ in which different, more complex problems of cleavage were examined in the classical formulation of the present section.

During cleavage of a body by a smooth rigid wedge with a dead zone under compression conditions (Fig.5), the potentials $\Phi(z)$ and $\Psi(z)$ as well as the stresses in the non-stationary rough-working process will have a power-law singularity at the wedge angles (for $z = 0$). In this case, in addition to the frontal resistance force from the fragements, a configurational resistance force component will also act on the wedge. By using the invariant Γ-integral it can be shown that this component equals

$$T = (1 - \nu^2) E^{-1} K_{1c}^2 + 2h\,(q - p) + 2\tau \Delta u \tag{5.7}$$
$$(\Delta u = u\,(0, 0) - u\,(l, 0))$$

During cleavage of a body by a smooth rigid wedge with a dead zone under tension conditions (see Fig.6 and Sect.4), the configurational resistance force component has the form

$$T = \Gamma_c - (1 - \nu^2)\, E^{-1} (K_1^\infty)^2 - 2hp + 2\tau \Delta u \tag{5.8}$$
$$(\Gamma_c = (1 - \nu^2) E^{-1} K_{1c}^2,\ \Delta u = u\,(0, 0) - u\,(l, 0))$$

In both cases the force T vanishes in the stationary stage.

As follows from published research, when calculating the resistance force in contact problems concerning wedge or stamp motion /11—15/, the configurational component of this force, generated when bypassing the singularities on the contact area, must be taken into account. In particular, when slicing an elastic body with a very thin blade, whose ends penetrate into the very tip of the forward-proceeding crack, the force of the resistance to its motion will equal the critical force Γ_c diminished by the configurational force from the external tension field

$$T = \Gamma_c - \frac{1 - \nu^2}{E} K_1^2$$

An analogous consequence holds with respect to the cutting work; the work of the external forces expended in the cutter motion diminishes by the magnitude of the work of the configurational force for $K_1 > 0$.

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